Function discussion

FD1

T: f: (a,b)   R

If $f'(x) \geq 0$ on an interval (a,b) $\Rightarrow$ f is monotone increasing on (a,b),

if $f'(x) > 0$ on an interval (a,b) $\Rightarrow$ f is strictly monotone increasing on (a,b),

if $f'(x) \leq 0$ on an interval (a,b) $\Rightarrow$ f is monotone decreasing on (a,b),

if $f'(x) < 0$ on an interval (a,b) $\Rightarrow$ f is strictly monotone decreasing on (a,b),

Example:

$f(x) = 2x^3 - 3x^2 - 12x + 3$

On which open intervals is the function increasing, decreasing?

$f'(x) = 6x^2 - 6x - 12 = 6\,(x^2 - x - 2) = 6\,(x-2)(x+1)$

Thus f is increasing if $6\,(x-2)(x+1) \geq 0$ and

f is decreasing if $6\,(x-2)(x+1) \leq 0$.

f is not increasing and decreasing as $6\,(x-2)(x+1) = 0$

	$x<-1$	$x=-1$	$-1<x<2$	$x=2$	$2<x$
$f'(x)$	+	0	−	0	+
$f(x)$	↑	local max.	↓	local min	↑

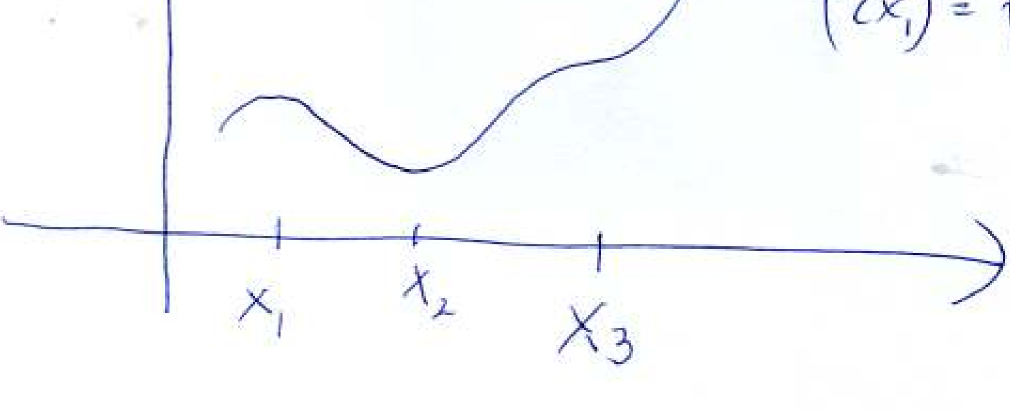
$f(-1) = 10$ $\qquad\qquad$ $f(2) = -17$

There is a local maximum of f at x = -1 with function value 10, And there is a local minimum of f at x = 2 with function value -17.

As $f'(x_0)=0$ the curve of f neither rises nor falls at point x_0. The function remains stationary, x_0 it is a <u>stationary point</u>.

<u>D</u>: **f: R   R ,** $\quad$ **$a \in D_f$,** $\quad$ **f is differentiable at a.**

If $\quad f'(a)=0 \quad \Rightarrow$ **a is a <u>stationary point</u>.**



$f'(x_1) = f'(x_2) = f'(x_3) = 0$

x_1, x_2, x_3 are stationary points.

At point x_1 the function reaches a local maximum value, that is in the neighboring points f(x) takes on smaller values.

At point x_2 the function reaches a local minimum value, that is in the neighboring points f(x) takes on greater values.

At point x_3 we have neither a maximum nor a minimum point.

T: $f:R \to R$ $\quad a \in D_f$
f is differentiable at point a
f' is differentiable at point a

At point a there is a maximum point if

$f'(a)=0$ (necessary condition) and
$f''(a)<0$ $\Leftrightarrow$ f' changes sign at point a, $f'(x)>0$ if $x<a$ and $f'(x)<0$ if $x>a$ (sufficient condition).

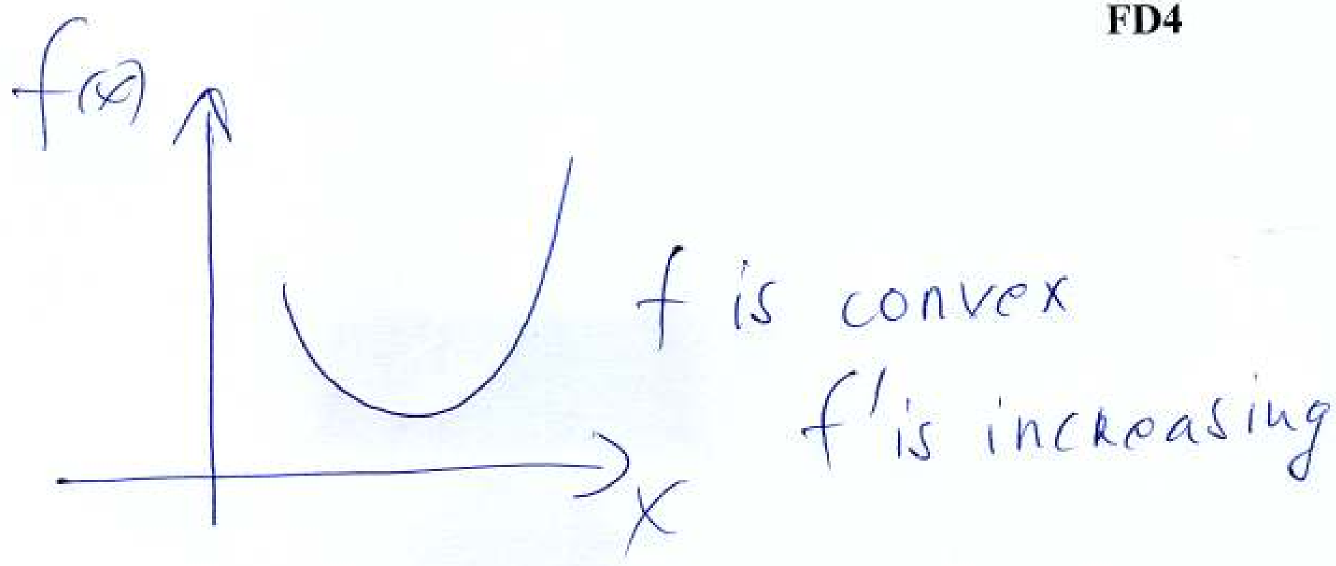
a is a minimum point if

$f'(a)=0$ (necessary condition) and
$f''(a)>0$ $\Leftrightarrow$ f' changes sign at point a, $f'(x)<0$ if $x<a$ and $f'(x)>0$ if $x>a$ (sufficient condition).

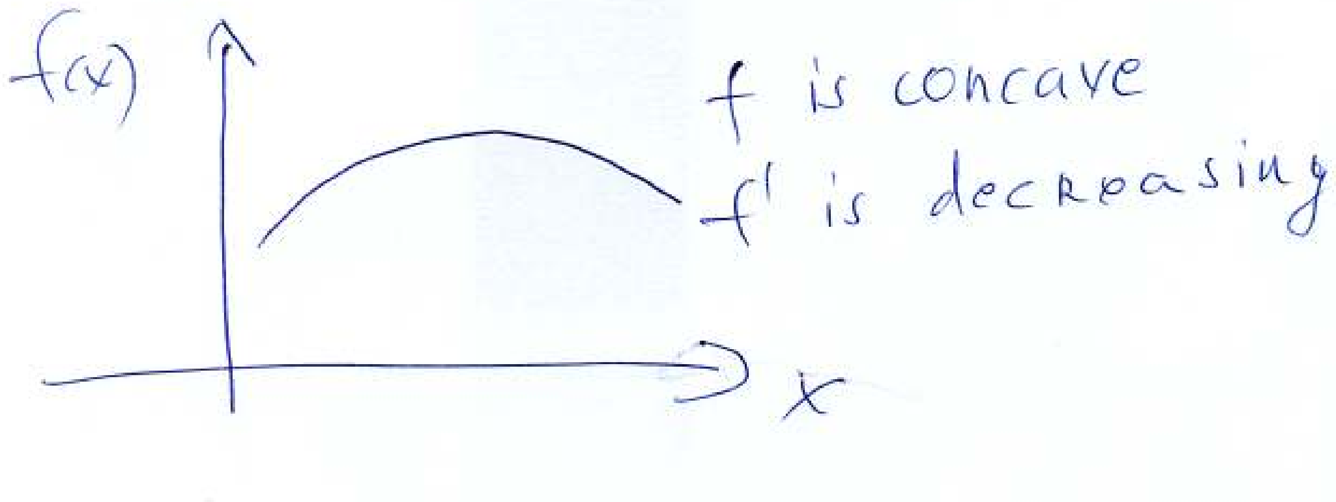
Intervals of convexity and intervals of concavity for $f:R \to R$

We may be interested in knowing whether the slopes of the tangents to the graph of f(x) are getting bigger or smaller with increasing x.

D: A function is called convex (convex down) in an interval I if f' is increasing. $\quad I \in D_f$



D: A function is **concave (concave down)** on an interval I if f'(x) is degreasing.



T: If $f''(x)>0$ on an interval I $\Rightarrow$ f(x) is convex on I.

T: If $f''(x)<0$ on an interval I $\Rightarrow$ f(x) is concave on I.

D: If f(x) is continuous at a point x_0 and changes from convex to concave or from concave to convex $\Rightarrow$ x_0 is an **inflection point**.

T: $f: R \to R$ $x_0 \in D_f$ f, f' and f'' are differentiable at x_0

x_0 is an inflection point of $f(x)$ if

$f''(x_0)=0$ (necessary condition) and

$f'''(x_0)>0$ or $f'''(x_0)<0$ $\Leftrightarrow$ $f''(x)$ changes sign at x_0 (sufficient condition).

Example: $f(x) = x^3 - 3x^2 + 3x + 1$

$f'(x) = 3x^2 - 6x + 3$

$f''(x) = 6x - 6$

$0 = 6x - 6$

$0 = \cancel{6}(x-1)$

$x = 1$

x=1 is a ~~stationary point and~~ a possible inflection point.

$f'''(x) = 6 > 0$

$\Longrightarrow$

X=1 is an inflection point.

Function discussion of f(x)

1. $D_f = ?$

2. $f(x)=0 \qquad x=?$

3. Is the function odd, even or periodic?

4. Find the limiting values of f(x) at the borders of D_f.

5. Discussion of $f'(x)$. Find the minimum, maximum, stationary points of f(x), increasing and degreasing intervals (if there is any).

6. Discussion of $f''(x)$. Find the inflection point/s, convex and concave intervals (if there is any).

7. Draw the graph of f(x) and find R_f.

Example: $f(x) = xe^{-x}$

1. $D_f = \mathbb{R}$

2. $f(x) = 0 \qquad 0 = xe^{-x} \qquad x = 0$

   $f(0) = 0$

3. non

4. $\lim\limits_{x \to +\infty} xe^{-x} = \lim\limits_{x \to +\infty} \frac{x}{e^x}$

   $\lim\limits_{x \to +\infty} x = \lim\limits_{x \to +\infty} e^x = +\infty$

$$\lim_{x\to\infty}\frac{1}{e^x}=0 \Longrightarrow \lim_{x\to+\infty} x e^{-x}=0$$ **FD7**

$$\lim_{x\to-\infty} x e^{-x}=-\infty$$

5. $f'(x)=e^{-x}+x(-1)e^{-x}=e^{-x}(1-x)$

$f'(x)=0 \qquad 0=e^{-x}(1-x)$

$$x=1$$

	$x<1$	$x=1$	$1<x$
$f'(x)$	$+$	0	$-$
$f(x)$	$\uparrow$	$f(1)=$ $=\frac{1}{e}$	$\downarrow$
		local max. point	

6. $f''(x)=-e^{-x}(1-x)+e^{-x}(-1)=$

$=-e^{-x}+xe^{-x}-e^{-x}=xe^{-x}-2e^{-x}=$

$=e^{-x}(x-2)$

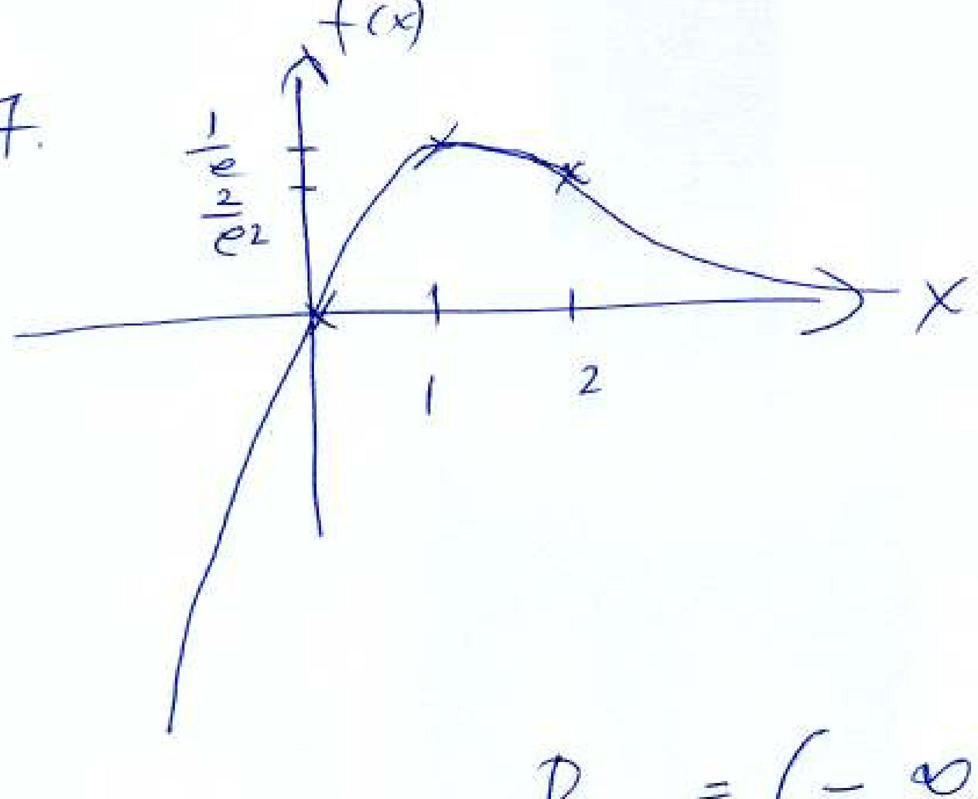
FD8

$f''(x) = 0 \qquad 0 = e^{-x}(x-2) \qquad x = 2$

	$x<2$	$x=2$	$2<x$
$f''(x)$	$-$	0	$+$
$f(x)$	$\frown$	infl. point	$\smile$

$f(2) = 2e^{-2} = \frac{2}{e^2}$

7.



$R_f = \left(-\infty, +\frac{1}{e}\right]$