# Application of differentiation

**T: The L'Hospital rule for** $\lim_{x \to c} \frac{f(x)}{g(x)}$

**f, f', g, g', f/g and f'/g' are defined on intervals (a,c) and (c,d) where** $a \angle c \angle b$

$\lim_{x \to c} f(x) = 0$

$\lim_{x \to c} g(x) = 0$ **and**

$$\exists \lim_{x \to c} \frac{f'(x)}{g'(x)} \quad \Rightarrow \quad \exists \lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

**Examples:**

1. $\lim_{x \to 4} \frac{x^2-16}{x-4} = \lim_{x \to 4} \frac{2x}{1} = 8$

$\lim_{x \to 4} x^2 - 16 = 0$

$\lim_{x \to 4} x - 4 = 0$

2. $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$

$\lim_{x \to 0} \sin x = 0 \qquad \lim_{x \to 0} x = 0$

3. $\lim\limits_{x \to 2} \frac{\ln x - \ln 2}{x - 2} = \lim\limits_{x \to 2} \frac{\frac{1}{x}}{1} = \frac{1}{2}$

$\lim\limits_{x \to 2} (\ln x - \ln 2) = 0 \qquad \lim\limits_{x \to 2} (x - 2) = 0$

**The L'Hospital rule is valid if c is infinity** (+, - $\infty$)

4. $\lim\limits_{x \to +\infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = \lim\limits_{x \to +\infty} \frac{\cancel{-x^{-2}} \cos \frac{1}{x}}{\cancel{-x^{-2}}} = 1$

$\lim\limits_{x \to +\infty} \sin \frac{1}{x} = 0 \quad , \lim\limits_{x \to +\infty} \frac{1}{x} = 0$

**T: The L' Hospital rule for** $\lim_{x \to c} \frac{f(x)}{g(x)}$

$\lim_{x \to c +- \infty}$ **f(x) =+-** $\infty$

$\lim_{x \to c +- \infty}$ **g(x)=+-** $\infty$ **and**

$\exists \lim_{x \to c +- \infty} \frac{f'(x)}{g'(x)} \quad \Rightarrow \quad \exists \lim_{x \to c +- \infty} \frac{f(x)}{g(x)} = \lim_{x \to c +- \infty} \frac{f'(x)}{g'(x)}$

**Examples:**

5. $\lim\limits_{x \to +\infty} \frac{5x^2 + 2x + 1}{3x + 1} = \lim\limits_{x \to +\infty} \frac{10x + 2}{3} = +\infty$

$\frac{\infty}{\infty}$

6. $\lim_{x \to +\infty} \frac{e^x}{2x} = \lim_{x \to +\infty} \frac{e^x}{2} = +\infty$

$\frac{\infty}{\infty}$

For many functions y= f(x) it is a practical problem <u>to find numerical values of y given corresponding values of x.</u>

Sometimes very rough approximation will do,
Sometimes accurate figures are required.

<u>**Approximation of f(x) by polynomial in x**</u>

<u>**Polynomial of first degree**</u>

y=f(x) is differentiable if this limit exists and is finite

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x)$$

So for a differentiable function f at a point x,

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x) + \varepsilon$$

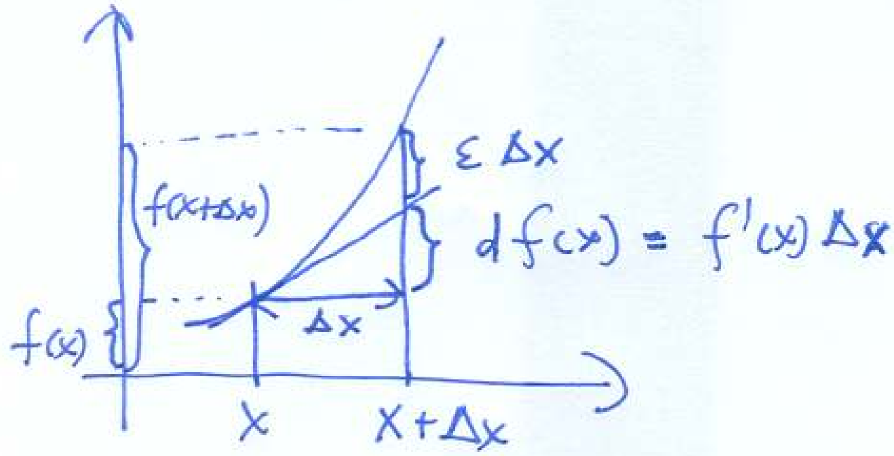
Multiplying the equation by $\Delta$x

$$\Delta f = f(x+\Delta x) - f(x) = f'(x)\,\Delta x + \varepsilon\,\Delta x$$

where $\varepsilon \to 0$ as $\Delta x \to 0$

**$\Delta$ f's principal part of increment is $f'(x)\,\Delta x$**

**$f'(x)\,\Delta x$ is denoted by $df(x)= f'(x)\,\Delta x$ and is called <u>the differential of function f(x) at the point x.</u>**




$$\Delta f = df(x) + \varepsilon \Delta x$$

**Example:**

Approximate cos $48^0$, cos $45^0 = \sqrt{2}/2$

$$f(48^\circ) = \cos 48^\circ \cong f(45^\circ) + f'(45^\circ)\,\Delta x =$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,\frac{3\cdot 2\cdot \pi}{360} \cong 0{,}6681$$

$$f = \cos x \quad f' = -\sin x \quad f'(45^\circ) = -\frac{\sqrt{2}}{2}$$

$$\Delta x = 48^\circ - 45^\circ = 3^\circ = 3\cdot\frac{2\pi}{360} \text{ radian}$$

Calculators value:

$\cos 48° = 0,6691$

**If better approximation is required __Taylor series__ are used.**

**__Series__**

**__D:__** a: **N** $\longrightarrow$ **R**

$$a_1 + a_2 + a_3 + \quad = \sum_{k=1}^{\infty} a_k$$

**let us define the __sequence of partial sums__**

**__D:__** a: **N** $\longrightarrow$ **R**

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$

$$s_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{k=1}^{n} a_k$$

**__D:__** **If the** s: **N** $\longrightarrow$ **R** **sequence of partial sums is convergent**

$$\lim_{n\to\infty} s = A$$

**then we say** $\sum_{k=1}^{\infty} a_k = A$

**If s: N$\to$R is not convergent we say the** $\sum_{k=1}^{\infty} a_k$ **series is divergent.**

**Example:**

**0.9 + 0.09 + 0 009 + 0.0009…..= 1**

**the series is convergent.**

**<u>Taylor series if f(x)</u>**

**n! = 1*2 *3* 4*…*n**

$$f(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \dots + \frac{f^{(n)}(a)(x-a)^n}{n!} + \dots =$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)(x-a)^k}{k!} \qquad 0! = 1$$

**If a = 0**

$$f(x) = f(0) + \frac{f'(0)\,x}{1!} + \frac{f''(0)\,x^2}{2!} + \dots + \frac{f^{(n)}(0)\,x^n}{n!} + \dots = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)\,x^k}{k!}$$

**It is customary to call the procedure as <u>f(x)' expansion into a series around center a</u>**

**Does** $$T_n(f(x)) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!}$$

**converge to f(x) as n tends to infinity?**

**$T_n$ is convergent, if there is an $r > 0$ such that for $\forall x \in (a-r, a+r)$**

**$\forall f(x)^{(i)} \leq K$, then $T_n$ is convergent for $\forall x \in (a-r, a+r)$**

**Examples:**

① $f(x) = e^x \qquad a = 0$

$f(x) = e^x \qquad f(0) = e^0 = 1$

$f'(x) = e^x \qquad f'(0) = 1$

$f''(x) = e^x \qquad f''(0) = 1$

$\vdots$

$f^{(n)}(x) = e^x \qquad f^{(n)}(0) = 1$

$$e^x = 1 + \frac{1x}{1!} + \frac{1x^2}{2!} + \frac{1x^3}{3!} + \ldots =$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

AD8

2. $f(x) = \sin x \qquad a = 0$

$f(x) = \sin x \qquad f(0) = \sin 0 = 0$

$f'(x) = \cos x \qquad f'(0) = \cos 0 = 1$

$f''(x) = -\sin x \qquad f''(0) = -\sin 0 = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -\cos 0 = -1$

$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = \sin 0 = 0$

$f^{(5)}(x) = \cos x \qquad f^{(5)}(0) = \cos 0 = 1$

$\vdots$

$$\sin x = 0 + \frac{1\,x}{1!} + \frac{0\cdot x^2}{2!} + \frac{(-1)\,x^3}{3!} +$$

$$+ \frac{0\cdot x^4}{4!} + \frac{1\cdot x^5}{5!} \cdots =$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots\cdots$$

3. $f(x) = \ln x$ $\quad a = 1$ $\quad$ ADG

$f(x) = \ln x$ $\qquad f(1) = \ln 1 = 0$

$f'(x) = \frac{1}{x} = x^{-1}$ $\qquad f'(1) = 1$

$f''(x) = (-1)x^{-2} = -\frac{1}{x^2}$ $\qquad f''(1) = -1$

$f'''(x) = (-1)(-2)x^{-3} = \frac{2}{x^3}$ $\qquad f'''(1) = 2$

$f^{(4)}(x) = 2(-3)x^{-4} = -\frac{6}{x^4}$ $\qquad f^{(4)}(1) = -6$

$f^{(5)}(x) = (-6)(-4)\cdot(x^{-5}) = \frac{24}{x^5}$ $\qquad f^{(5)}(1) = 24$

$\vdots$

$$\ln x = 0 + \frac{1\,(x-1)}{1!} + \frac{(-1)(x-1)^2}{2!} + \frac{2\,(x-1)^3}{3!} + \frac{(-6)\,(x-1)^4}{4!} + \frac{24\,(x-1)^5}{5!} + \cdots$$