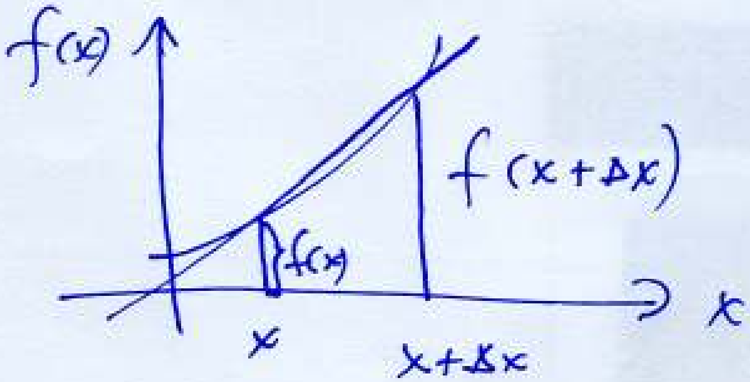
Differentiation

The derivative of **f(x): R→R**



The slope of the secant line is:

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Let us take the limit:

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

The function f(x) is differentiable at the point $x \in D_f$ if this limit exists and is a finite one.

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x) = \frac{df(x)}{dx}$$

$f'(x)$ is called the function's derivative at point x.

$f'(x)$ (if it exists) gives the slope of the tangent line to the graph of $f(x)$ at the point $(x,y) = (x,f(x))$

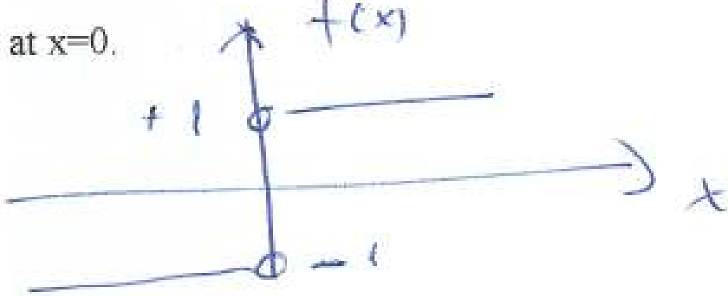
f fails to be differentiable at a point $x \in D_f$ if it has a break, a corner or a vertical tangent line at this point.

Examples:

f_1, f_2 and f_3 are not differentiable at $x=0$.

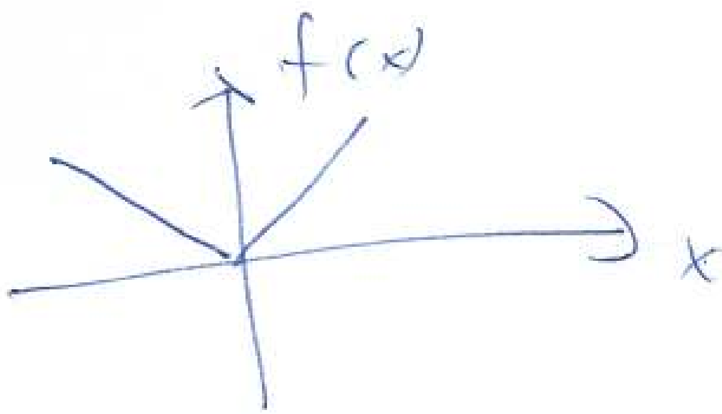
1. $f_1(x) = \frac{x}{|x|}$

$0 \notin D_f$



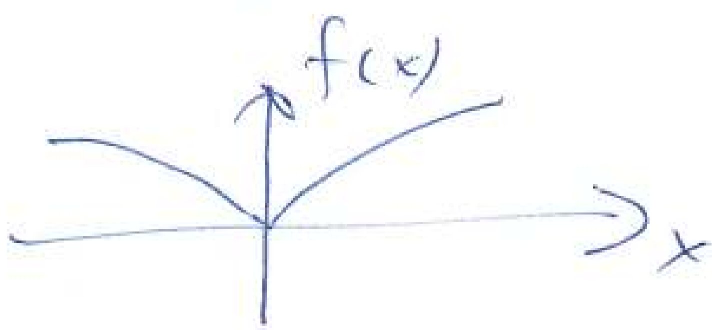
2. $f_2(x) = |x|$

$0 \in D_f$



3. $f_3(x) = \sqrt{|x|}$

$0 \in D_f$



T: **If f(x) is differentiable at $x \Rightarrow$ f(x) is continuous at x.**
The converse of the statement is not true.

Techniques of differentiation

Instead of obtaining derivatives directly from the definition techniques and formulae were made to differentiate functions.
/Most of the proofs: see appendix/

T:. $\mathbf{c' = 0} \qquad \mathbf{c \in R}$

Proof:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{c - c}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{0}{\Delta x} = \lim_{\Delta x \to 0} 0 = 0$$

T: **If f and g are differentiable at the point x then f+g and f-g are also differentiable at the point x and**

$\mathbf{(f \pm g)' = f' \pm g'}$

Proof:

$$(f \pm g)' = \lim_{\Delta x \to 0} \frac{\left(f(x+\Delta x) \pm g(x+\Delta x)\right) - \left(f(x) \pm g(x)\right)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{\left(f(x+\Delta x) - f(x)\right) \pm \left(g(x+\Delta x) - g(x)\right)}{\Delta x} =$$

$$= f'(x) \pm g'(x)$$

Example:

$$(5x^3 - 3x^2)' = 5 \cdot 3 \cdot x^2 - 3 \cdot 2x$$

T: **If f is differentiable at x and $c \in R$, then cf is also differentiable at x and**

(cf)' = cf'

Proof:

$$(c f(x))' = \lim_{\Delta x \to 0} \frac{c f(x+\Delta x) - c f(x)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} c \, \frac{f(x+\Delta x) - f(x)}{\Delta x} = c \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \, .$$

$$= c \, f'(x)$$

Example:

$$(5x^3)' = 5 \cdot (x^3)' = 5 \cdot 3 x^2$$

T: **If f and g are differentiable at the point x, then fg is also differentiable at x and**

(fg)' = f'g + f g'

Proof. see appendix.

Example:

$$(5x^2 e^x)' = (5x^2)' e^x + 5x^2 (e^x)' =$$

$$= 10 x e^x + 5x^2 e^x$$

T: $(x^n)' = n \, x^{n-1}$ $\quad n \in R$

Proof. see appendix.

Example:

$$\left(5\sqrt{x} + 2x^2 + \frac{1}{\sqrt{x}}\right)' = \left(5x^{\frac{1}{2}} + 2x^2 + x^{-1/2}\right)' =$$

$$= 5 \cdot \tfrac{1}{2} x^{-\frac{1}{2}} + 2 \cdot 2x - \tfrac{1}{2} x^{-3/2}$$

T: **If f and g are differentiable at the point x, then f/g is also differentiable at x and**

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Proof. see appendix.

Example.

$$\left(\frac{e^x}{x^2+x}\right)' = \frac{e^x(x^2+x) - e^x(2x+1)}{(x^2+x)^2}$$

T. $(e^x)' = e^x$ $\quad$ $(a^x)' = a^x \ln a$

Proofs. see appendix

T: $\ln x' = 1/x$ $\quad$ $\log_a x' = 1/(\ln a)\, x$

Proofs. see appendix

T: $\sin x' = \cos x$

$\cos x' = -\sin x$

$\tan x' = 1/\cos^2 x$

$\text{cotan}\, x' = -1/\sin^2 x$

Proofs. see appendix

T: $\arcsin x' = \frac{1}{\sqrt{1-x^2}}$

$\arccos x' = -\frac{1}{\sqrt{1-x^2}}$

$\arctan x' = \frac{1}{1+x^2}$

$\operatorname{arccotan} x' = -\frac{1}{1+x^2}$

Proofs. see appendix

T: **<u>The chain rule</u>**

If g is differentiable at x and f is differentiable at g(x) then f(g(x)) is also differentiable at x and

$f(g(x))' = f'(g(x)) \; g'(x)$

Proof:

$$f'(x) = \big(f(g(x))\big)' = \lim_{\Delta x \to 0} \frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{f(g(x+\Delta x)) - f(g(x))}{g(x+\Delta x) - g(x)} \cdot \frac{g(x+\Delta x) - g(x)}{\Delta x} =$$

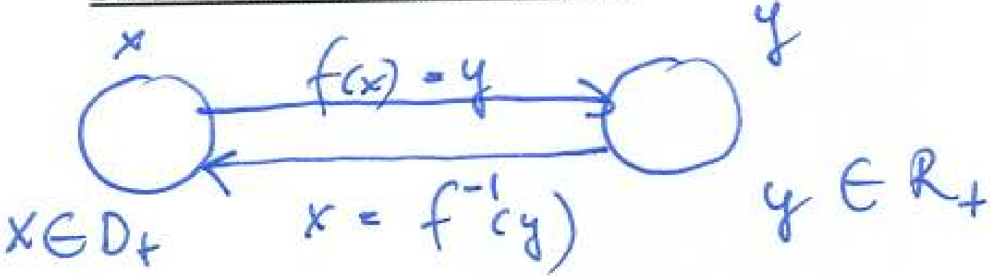
$$= f'(g(x)) \cdot g'(x)$$

Example.

① $$\left(\sin(x^2+x+1)\right)' =$$
$$= \left(\cos(x^2+x+1)\right)(2x+1)$$

② $$\left(\ln(2x^3+2x+2)\right)' =$$
$$= \frac{1}{2x^3+2x+2}\cdot(6x^2+2)$$

<u>Differentiation of inverse functions</u>



$$f^{-1}(f(x)) = x \qquad x \in D_f$$

$${f^{-1}}'(f(x))\cdot f'(x) = 1$$

$$f'(x) = \frac{1}{f^{-1\prime}(f(x))}$$

Example:

① $f(x) = \ln x = y$

$f^{-1}(y) = e^{y} = x$

$f'(x) = \ln x' = \frac{1}{e^{y}} = \frac{1}{x}$

② $f(x) = \sqrt[n]{x} = x^{\frac{1}{n}}$ $\quad n \in N$

$f(x) = x^{\frac{1}{n}} = y$

$f^{-1}(y) = y^{n}$

$f'(x) = \left(x^{\frac{1}{n}}\right)' = \frac{1}{n\, y^{n-1}} =$

$= \frac{1}{n} \cdot \frac{1}{x^{\frac{1}{n}(n-1)}} = \frac{1}{n} \cdot \frac{1}{x^{1-\frac{1}{n}}} =$

$= \frac{1}{n} \cdot \frac{1}{x^{\frac{n-1}{n}}} = \frac{1}{n} x^{-\frac{n-1}{n}} =$

$= \frac{1}{n} x^{\frac{1}{n}-1}$

<u>**Differentiation of $f(x)^{g(x)}$ type functions**</u>

$$f(x)^{g(x)} = e^{\ln f(x)^{g(x)}} =$$

$$= e^{g(x)\ln(f(x))} \qquad f(x) > 0$$

$$\left(f(x)^{g(x)}\right)' = \left(e^{g(x)\ln(f(x))}\right)' =$$

$$= e^{g(x)\ln f(x)}\left(g'(x)\ln(f(x)) + \right.$$

$$\left. + g(x)\frac{1}{f(x)}f'(x)\right)$$

Higher derivatives

D: **Let f be differentiable at some interval. f' is the derivative or first derivative of f.**

If f is differentiable we denote it's derivative f'' (second derivative)

$$f''(x) = (f'(x))'$$

As long as we have differentiability, we can continue differentiating derivatives to obtain third, fourth, fifth and even higher derivatives of f.

The successive derivatives of f are denoted with.

$$f''(x) = (f'(x))', \quad f'''(x) = (f''(x))',$$

$$f^{(4)}(x) = (f'''(x))' \quad \ldots\ldots$$

D: **If $f^{(n-1)}$ is differentiable then its derivative ($n \in N$)**

$(f^{(n-1)})' = f^{(n)}$ is the n-th derivative of f.

Examples:

$$f(x) = 3x^2 + 2x + 1$$
$$f'(x) = 6x + 2$$
$$f''(x) = 6$$
$$f'''(x) = 0$$

$$f^{(4)}(x) = 0$$
$$f^{(5)}(x) = 0$$
$$\ldots$$

Appendix Proofs

D11

① $(fg)' = f'g + fg'$

$$(fg)' = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) \cdot g(x+\Delta x) - f(x) \cdot g(x)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)\, g(x+\Delta x) + f(x) g(x+\Delta x) - f(x)\, g(x)}{\Delta x} =$$

$$= \lim_{\Delta x \to} g(x+\Delta x) \frac{f(x+\Delta x) - f(x)}{\Delta x} + f(x) \frac{g(x+\Delta x) - g(x)}{\Delta x} =$$

$$= f'(x)\, g(x) + f(x)\, g'(x)$$

② $(x^n)' = n x^{n-1}$ D12

a) $n \in N$

By using the identity

$$(a^n - b^n) = (a-b)(a^{n-1} + a^{n-2}b + \dots + ab^{n-2} + b^{n-1})$$

$$\lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x - x)((x+\Delta x)^{n-1} + (x+\Delta x)^{n-2} \cdot \Delta x + \dots + (x+\Delta x)\Delta x^{n-2} + \Delta x^{n-1})}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}}{\cancel{\Delta x}}((x+\Delta x)^{n-1} + (x+\Delta x)^{n-2}\Delta x + \dots + (x+\Delta x)\Delta x^{n-2} + \Delta x^{n-1})$$

$$= n x^{n-1}$$

b) $\left(x^{\frac{1}{n}}\right)' = \frac{1}{n} x^{\frac{1}{n}-1} \quad n \in \mathbb{N}$ D13

$$f(x) = \sqrt[n]{x} = x^{\frac{1}{n}} = y$$

$$f^{-1}(y) = y^n$$

$$f'(x) = \frac{1}{{f^{-1}}'(f(x))} = \frac{1}{n y^{n-1}} =$$

$$= \frac{1}{n\, x^{\left(\frac{1}{n}\right) n-1}} = \frac{1}{n} X^{\frac{1}{n}-1}$$

3 $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) g(x) - f(x) g'(x)}{g^2(x)}.$

$$\left(\frac{f(x)}{g(x)}\right)' = \lim_{\Delta x \to 0} \frac{\frac{f(x+\Delta x)}{g(x+\Delta x)} - \frac{f(x)}{g(x)}}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)\, g(x) - f(x)\, g(x+\Delta x)}{g(x+\Delta x)\, g(x)\, \Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)\, g(x) - f(x)\, g(x) + f(x)\, g(x) - f(x)\, g(x+\Delta x)}{g(x+\Delta x)\, g(x)\, \Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{g(x)\left(f(x+\Delta x) - f(x)\right)}{g(x)\, g(x+\Delta x)\, \Delta x} - \lim_{\Delta x \to 0} \frac{f(x)\left(g(x+\Delta x) - g(x)\right)}{g(x)\, g(x+\Delta x)\, \Delta x} =$$

$$= \frac{f'(x)\, g(x)}{g(x)^2} - \frac{f(x)\, g'(x)}{g^2(x)} =$$

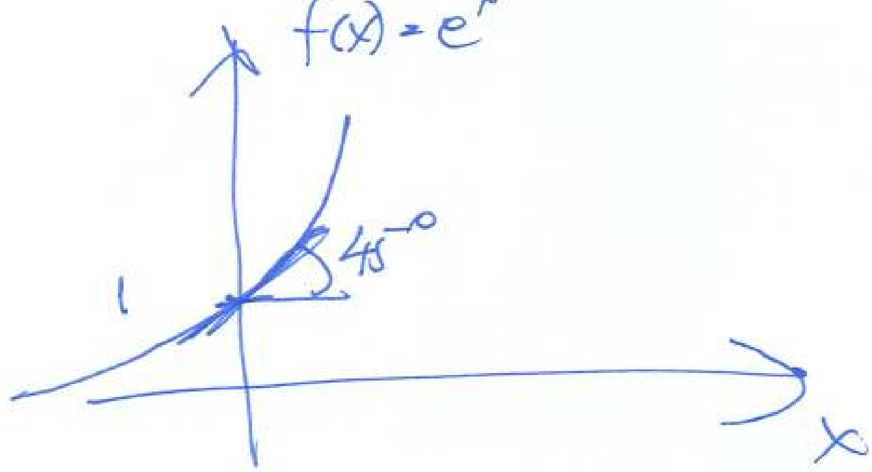
$$= \frac{f'(x)\, g(x) - f(x)\, g'(x)}{g^2(x)}$$

④ $(e^x)' = e^x$

$$(e^x)' = \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{e^x \cdot e^{\Delta x} - e^x}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} e^x \frac{e^{\Delta x} - 1}{\Delta x} = \lim_{\Delta x \to 0} e^x \frac{e^{\Delta x} - e^0}{\Delta x} =$$



at $x = 0$ the slope of the tangent line is 1.

$$= e^x$$

5. $(a^x)' = (\ln a)\, a^x$

$a^x = e^{x \ln a}$

$(a^x)' = e^{x \ln a} (\ln a) = a^x (\ln a)$

6. $(\log_a x)' = \dfrac{1}{(\ln a)\, x}$

$\log_a x = y \iff a^y = x$

$\ln a^y = \ln x$

$y \ln a = \ln x$

$\log_a x \, (\ln a) = \ln x$

$\log_a x = \dfrac{\ln x}{\ln a}$

$(\log_a x)' = \dfrac{1}{\ln a} (\ln x)' = \dfrac{1}{(\ln a) x}$

(7) $\sin' x = \cos x$

Proof: see next lecture

(8) $\cos' x = -\sin x$

$$\cos x = (1 - \sin^2 x)^{1/2}$$

$$\cos' x = \frac{1}{\cancel{2}}(1 - \sin^2 x)^{-1/2}(-\cancel{2}\sin x \cos x) =$$

$$= -\frac{\sin x \cancel{\cos x}}{\cancel{\cos x}} = -\sin x$$

(9) $\tan' x = \frac{1}{\cos^2 x}$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\tan' x = \frac{\cos x \cdot \cos x + \sin x \sin x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

(10) $$\cot an' x = -\frac{1}{\sin^2 x}$$

$$\cot an\, x = \frac{\cos x}{\sin x}$$

$$\cot an' x = \frac{-\sin x \cdot \sin x - \cos x \cos x}{\sin^2 x} =$$

$$= -\frac{1}{\sin^2 x}$$

(11) $$\arcsin' x = \frac{1}{\sqrt{1-x^2}}$$

$$f(x) = \arcsin x = y$$

$$\sin y = x \qquad f^{-1}(y) = x$$

$$f'(x) = \frac{1}{{f^{-1}}'(f(x))} = \frac{1}{\cos y} =$$

$$= \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

$$\cos^2 y = \frac{1}{1+\tan^2 y}$$

$$f'(x) = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

(14) $\operatorname{arc\,cotan}' x = -\frac{1}{1+x^2}$

proof is like in 13.