Limit of a function

Limit of a function is a fundamental concept in analyses.

D f: **R** → **R** we say the limit of f as x approaches a is A and write

$$\lim_a f(x) = A$$

if and only if for every $\varepsilon > 0$ there exist a $\delta > 0$ such that

$|f(x) - A| < \varepsilon$ $\quad (A - \varepsilon < f(x) < A + \varepsilon)$,

whenever

$0 < |x - a| < \delta$ $\quad (a - \delta < x < a + \delta, \quad x \neq 0)$

Note that f(a) need not be defined.

T $\lim_a f(x) = A \Leftrightarrow \forall$ x sequences ($x: \mathbf{N} \to \mathbf{R}$) with $\lim x = A$

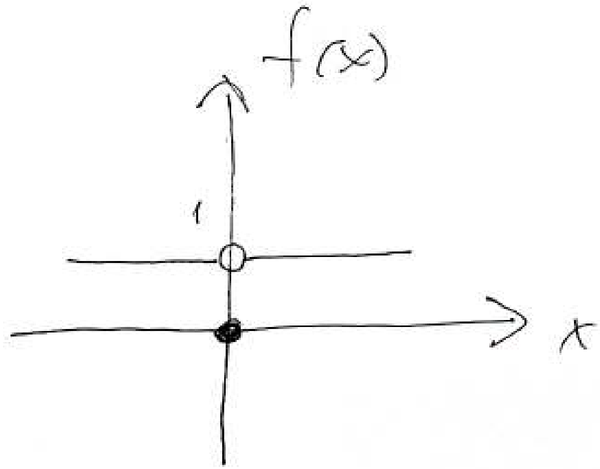
$(x_n \in D_f, x_n \neq a)$ $\quad \lim f(x_n) = A$

Examples

1. f: **R** → **R**

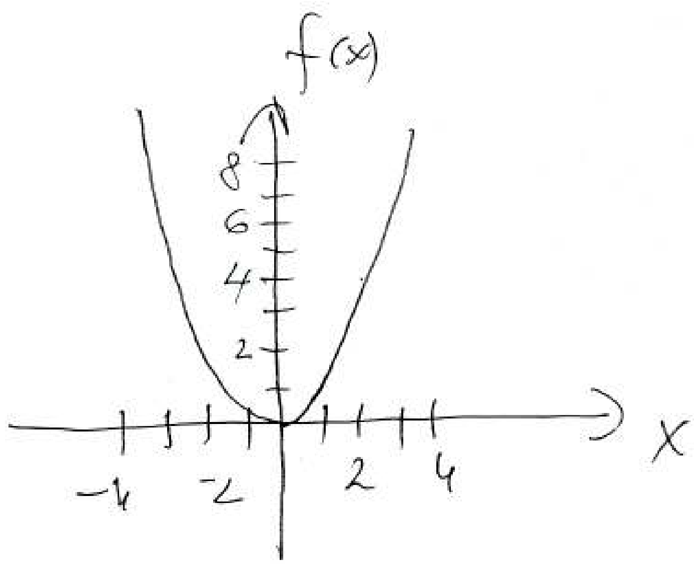
$$f(x) = \begin{cases} 1 \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

L2



$\lim_{0} f(x)=1$

2. f: **R** $\longrightarrow$ **R** **$f(x)= x^2$**



$\lim_{2} f(x)=4$

<u>D</u> f: **R** $\longrightarrow$ **R** It is said that <u>limit of f is positive infinity as x approaches a</u> ($\lim_{a} f(x) = +\infty$) if and only if for every $K > 0$ there is a $\delta > 0$ so that for all $x \in D_f$ and $0 < |x-a| < \delta$ $f(x) > K$

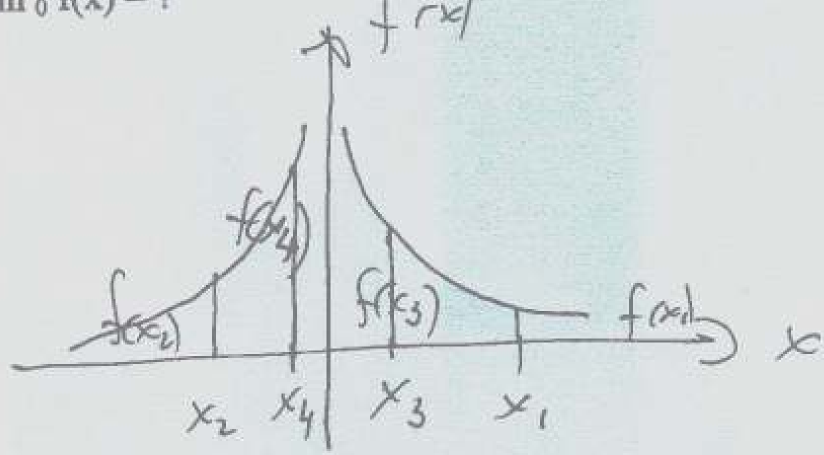
T $\lim_a f(x) = +\infty$ $\Leftrightarrow$ $\forall x$ sequences $(x: \mathbf{N} \to \mathbf{R})$ with $\lim x = a$

$(x_n \in D_f, x_n \neq a)$ $\lim f(x_n) = +\infty$

Examples

1. f: $\mathbf{R} \to \mathbf{R}$ $f(x) = 1/x^2$

$\lim_0 f(x) = ?$



x_1	$f(x_1)$
x_2	$f(x_2)$
x_3	$f(x_3)$
x_4	$f(x_4)$

$\lim x_n = 0$ $x_n \in D_f, x_n \neq a = 0$

$\lim f(x_n) = +\infty$ $\lim_0 f(x) = +\infty$

D f: **R** → **R** $\lim_a f(x) = -\infty$

if and only if for $\forall K < 0$ there is a $\delta > 0$ so that for $\forall x \in D_f$ and $0 < |x - a| < \delta$

$$f(x) < K$$

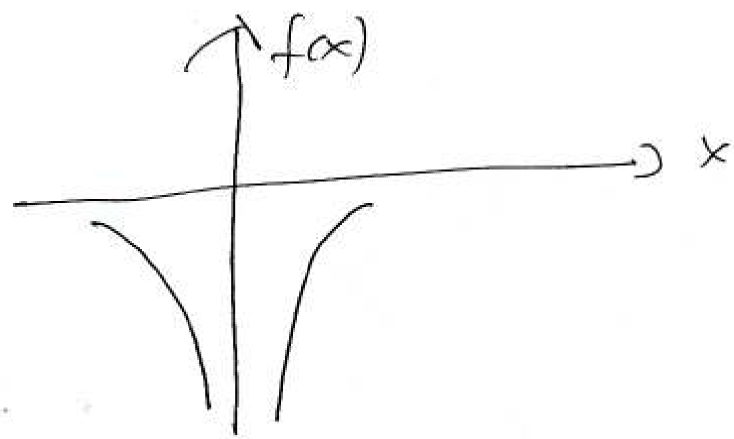
T f: **R** **R** $\lim_a f(x) = -\infty \Leftrightarrow \forall x$ sequences

(x: **N** → **R**) with $\lim x = a$

$(x_n \in D_f,\ x_n \neq a)$ $\lim f(x_n) = -\infty$

Example

f: **R** → **R** $f(x) = -1/x^2$



$\text{Lim}_0\, f(x) = -\infty$

D f: **R** → **R** $\lim_{a+0} f(x) = A$ if and only if

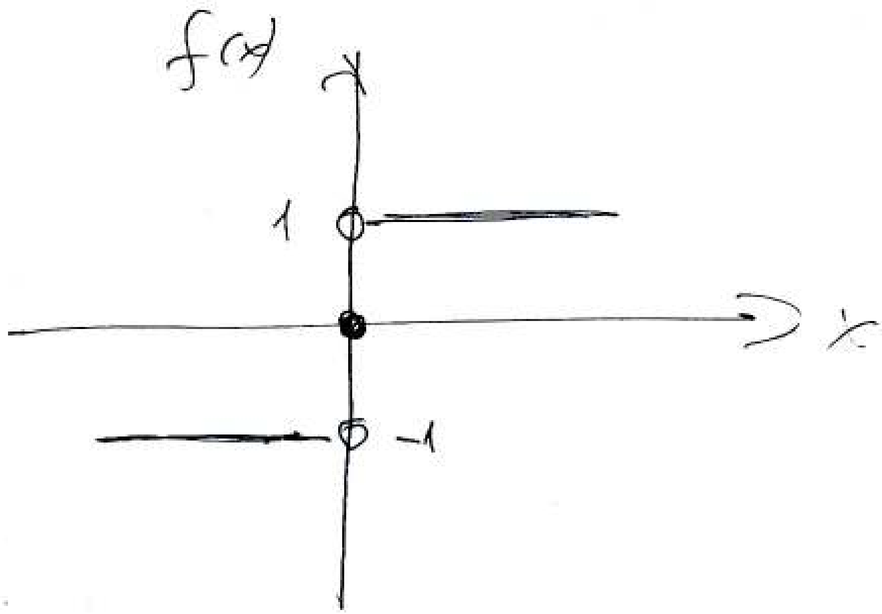
$\forall$ x sequences (x: **N** → **R**) with $\lim x = A$

$(x_n \in D_f, x_n \neq a, x_n \rangle a)$ $\lim f(x_n) = A$

Example

1. f: **R** → **R**

$$f(x) = \begin{cases} +1 & 0 \langle x \\ 0 & x = 0 \\ -1 & x \langle 0 \end{cases}$$



$\lim_{0+0} f(x) = 1$ $\lim_{0-0} f(x) = -1$

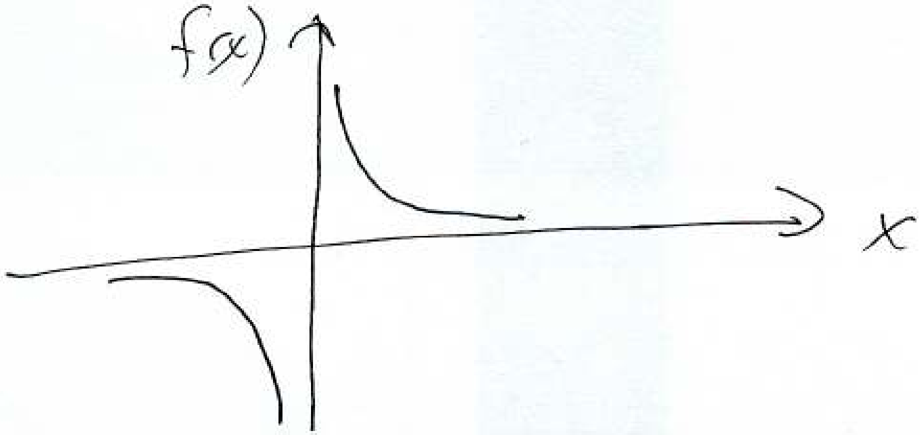
D f: **R** → **R** $\lim_{a-0} f(x) = A$ if and only if

$\forall$ x sequences (x: **N** → **R**) with $\lim x = a$

$(x_n \in D_f,\ x_n \neq a,\ x_n < a)$ $\lim f(x_n) = A$

Example

f: **R** → **R** $f(x) = 1/x$



$\lim_{0-0} f(x) = -\infty$ $\lim_{0+0} f(x) = +\infty$

$\nexists \lim_0 f(x)$ because $\lim_{0-0} f(x) = \lim_{0+0} f(x)$

D f: **R** → **R**

$\lim_{a-0} f = \pm\infty$ if and only if $\forall$ x sequences (x: **N** → **R**) with

$\lim x = a$ $(x_n \in D_f,\ x_n \neq a,\ x_n < a)$ $\lim f(x_n) = \pm\infty$

D f: $\mathbf{R} \to \mathbf{R}$

$\lim_{a+0} f = \pm\infty$ if and only if $\forall$ x sequences $(x: \mathbf{N} \to \mathbf{R})$ with

$\lim x = a$ $(x_n \in D_f,\ x_n \neq a,\ x_n > a)$ $\lim f(x_n) = \pm\infty$

T f: $\mathbf{R} \to \mathbf{R}$

$\lim_a f = A, +, -\infty \iff \lim_{a+0} f = \lim_{a-0} f = A, +, -\infty$

Operations and limits

T f: $\mathbf{R} \to \mathbf{R}$

g: $\mathbf{R} \to \mathbf{R}$

$\lim_{a,\ a+0,a-0} f = A,$ $\lim_{a,\ a+0,a-0} g = B$

then $\lim_{a,\ a+0,a-0} (cf) = cA$ $c \in \mathbf{R}$,

$\lim_{a,\ a+0,a-0} (f + g) = A + B,$

$\lim_{a,\ a+0,a-0} (f - g) = A - B,$

$\lim_{a,\ a+0,a-0} (f\,g) = A\,B,$

$\lim_{a,\ a+0,a-0} (f/g) = A/B$ $B \neq 0.$

Limit at + or - ∞

D f: $\mathbf{R} \to \mathbf{R}$

$\lim_{+,-\infty} f = A$ if and only if $\forall$ x sequences $(x: \mathbf{N} \to \mathbf{R})$ with

$\lim x = +,-\infty$ $(x_n \in D_f)$ $\lim f(x_n) = A$

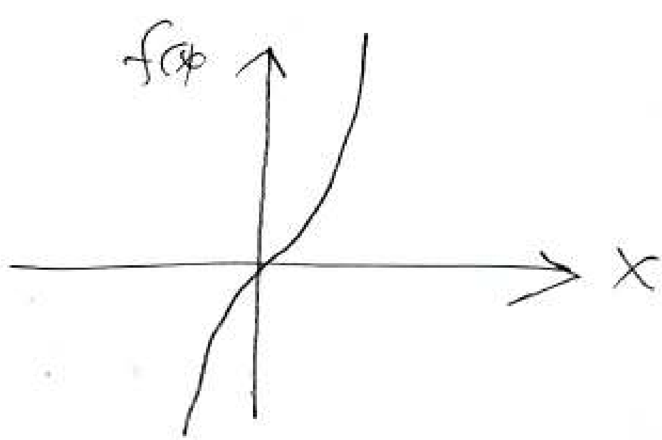
D f: $\mathbf{R} \to \mathbf{R}$

$\lim_{+,-\infty} f = +,-\infty$ if and only if $\forall$ x sequences $(x: \mathbf{N} \to \mathbf{R})$ with

$\lim x = +,-\infty$ $(x_n \in D_f)$ $\lim f(x_n) = +,-\infty$

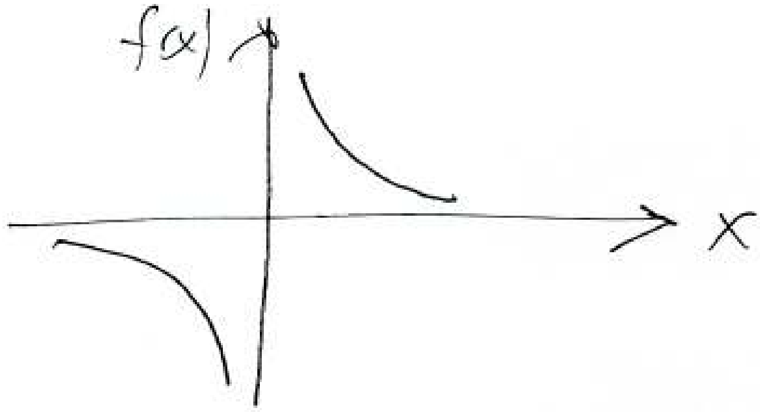
Examples

1. f: $\mathbf{R} \to \mathbf{R}$ $f(x) = x^3$



$\lim_{+\infty} f = +\infty$ $\lim_{-\infty} f = -\infty$

2. f: $\mathbf{R} \to \mathbf{R}$ $f(x) = 1/x$



$\lim_{+\infty} f = 0$, $\lim_{-\infty} f = 0$

The theorem for lim (cf), lim (f+g), lim (f-g), lim fg, lim f/g is valid in case of limits at + os - ∞.

Continuity is lack of interruption. Descartes definition: a function is continuous if its graph can be drawn without lifting the pencil from the paper.

<u>T</u> f: $\mathbf{R} \to \mathbf{R}$ is continuous at a point $a \in D_f$ if and only if

$f(a) = \lim_{a} f(x)$