Sequences

SE1

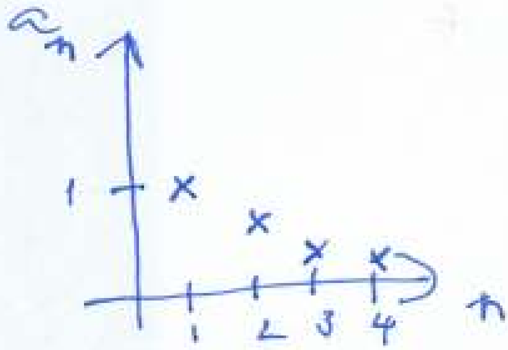
D $\quad a: \mathbf{N} \longrightarrow \mathbf{R}$

$a(n) = a_n$

examples:

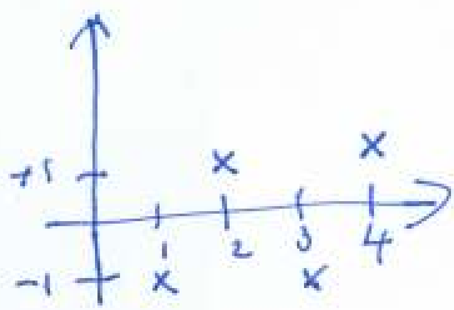
1. $a_n = 1/n$

$n=1$	$a_1=1$
$n=2$	$a_2=1/2$
$n=3$	$a_3=1/3$
$n=4$	$a_4=1/4$ ……



2. $a_n = (-1)^n$

$n=1$	$a_1=-1$
$n=2$	$a_2=1$
$n=3$	$a_3=-1$
$n=4$	$a_4=1$ ……



3. $a_n = 2$

4. $a_n = 2^n$

D a: **N** $\longrightarrow$ **R**

is <u>strictly monotone increasing</u> if for $\forall\, n \in \mathbf{N}$ $\uparrow$

$a_n < a_{n+1}$

is <u>monotone increasing</u> if for $\forall\, n \in \mathbf{N}$ $\nearrow$

$a_n \leq a_{n+1}$

is <u>strictly monotone degreasing</u> if for $\forall\, n \in \mathbf{N}$ $\downarrow$

$a_n > a_{n+1}$

is <u>monotone degreasing</u> if for $\forall\, n \in \mathbf{N}$ $\searrow$

$a_n \geq a_{n+1}$

D a: **N** $\longrightarrow$ **R** is a bounded sequence if $\exists\, K \in \mathbf{R}$
that for $\forall\, n \in \mathbb{N}$
$|a_n| \leq K$

In the previous examples,
sequence of example 1. is bounded, $\downarrow$
sequence of example 2. is bounded,
sequence of example 3. is bounded and $\nearrow, \searrow$
sequence of example 4. is not bounded and $\uparrow$.

In the theory of limit we are interested in the behavior of a sequence when n takes larger and larger values – in other words, when n tends to infinity.

D a: **N** $\longrightarrow$ **R** is said to tend or converge to a given value A, if for any $\varepsilon > 0$ is possible to find a natural number n_ε such that

$$|a_n - A| < \varepsilon$$

holds whenever $n \geq n_\varepsilon$.

in symbols: $\lim a_n = A$ (or $a_n \longrightarrow A$)

Sequences not converging to a finite value A are divergent sequences.

Examples:

1. a: **N** $\longrightarrow$ **R** $a_n = (-1)^n / n$

$a_1 = -1$
$a_2 = 1/2$
$a_3 = -1/3$
$a_4 = 1/4$
$a_5 = -1/5$
$a_6 = 1/6$

$\lim a_n = 0$

2. a: $\mathbf{N} \longrightarrow \mathbf{R}$ $a_n = 3/n$

$\lim a_n = 0$

Limit of some important sequences

$\lim c = c$ $c \in \mathbf{R}$

$\lim 1/n^k = 0$ if k is a + rational number

$\lim q^n = 0$ if $|q| < 1$

T a: $\mathbf{N} \longrightarrow \mathbf{R}$
b: $\mathbf{N} \longrightarrow \mathbf{R}$
a, b are convergent sequences, $\lim a = A$, $\lim b = B$

ca (c **R**), a+b, a-b, a b, a/b (b=0) are also convergent sequences and

$\lim ca = c A$
$\lim a+b = A+B$
$\lim a-b = A-B$
$\lim a b = A B$
$\lim a/b = A/B$ $(B \neq 0)$

Examples:

1. a: $\mathbf{N} \longrightarrow \mathbf{R}$

$$a_n = 3 + \frac{5}{2^n} = 3 + 5\left(\frac{1}{2}\right)^n$$

$\lim a = 3 + 5 \cdot 0 = 3$

2. a: $\mathbf{N} \longrightarrow \mathbf{R}$

$$a_n = \frac{1+2n}{4-n} = \frac{\cancel{n}\left(\frac{1}{n}+2\right)}{\cancel{n}\left(\frac{4}{n}-1\right)}$$

$\lim a = -2$

3. a: $\mathbf{N} \longrightarrow \mathbf{R}$

$$a_n = \frac{1+n+2n^2}{4-n^3} = \frac{\cancel{n^2}\left(\frac{1}{n^2}+\frac{1}{n}+2\right)}{\cancel{n^3}\left(\frac{4}{n^3}-1\right)}$$

$\lim a = 0$

<u>Divergent sequences</u>

<u>**D**</u> a: $\mathbf{N} \longrightarrow \mathbf{R}$. a is said to tend or to diverge to infinity, if for any given

$K >$ it is possible to find a natural number (n_K) such that

$\forall\, n \geq n_k \qquad a_n > K$

In symbols

$\lim a = +\infty$ (or $a_n \longrightarrow +\infty$)

<u>Example</u> a: $\mathbf{N} \longrightarrow \mathbf{R}$ $\quad a_n = 2^n$

$\lim a = +\infty$

2. a: **N** $\longrightarrow$ **R**

$$a_n = \frac{1+2n}{4-n} = \frac{\cancel{n}\left(\frac{1}{n}+2\right)}{\cancel{n}\left(\frac{4}{n}-1\right)}$$

lim $a = \frac{2}{-1} = -2$

3. a: **N** $\longrightarrow$ **R**

$$a_n = \frac{1+n+2n^2}{4-n^3} = \frac{\cancel{n^2}\left(\frac{1}{n^2}+\frac{1}{n}+2\right)}{n^{\cancel{3}}\left(\frac{4}{n^3}-1\right)}$$

lim a $= 0$

<u>Divergent sequences</u>

<u>D</u> a: **N** $\longrightarrow$ **R**. a is said to tend or to diverge to infinity, if for any given $K > 0$ it is possible to find a natural number such that (n_k)

$$\forall\, n \geq n_k \qquad a_n > K$$

In symbols

lim a $= +\infty$ (or $a_n \longrightarrow +\infty$)

<u>Example</u> a: **N** $\longrightarrow$ **R** $\quad$ $a_n = 2^n$

lim a $= +\infty$

D a: **N** $\longrightarrow$ **R** it is said to <u>diverge to the negative infinity</u>, if for any $K < 0$

It is possible to find a natural number (n_K) such that for

$$\forall n \geq n_K \qquad a_n < K$$

In symbols $\lim a = -\infty$ (or $a_n \longrightarrow -\infty$)

Examples

1. $a_n = -2^n$ $\lim a = -\infty$

2. $a_n = (-n)$ $\lim a = -\infty$

3.

$$a_n = \frac{n^3 + 2n + 1}{-n^2 + n + 1} = \frac{n^{\cancel{3}}\left(1 + \frac{2}{n^2} + \frac{1}{n^3}\right)}{\cancel{n^2}\left(-1 + \frac{1}{n} + \frac{1}{n^2}\right)}$$

$\lim a = -\infty$