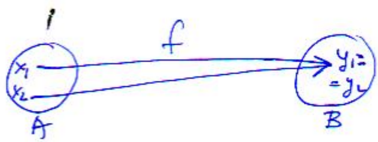
Function

$A, B \neq \phi$ $\quad f: A \longrightarrow B$ means to each $x \in A$ there is associated a unique $y \in B$.

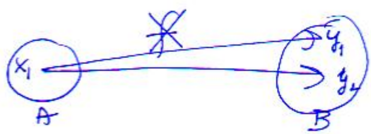
$x_1, x_2 \in A$ $\qquad\qquad$ $y_1, y_2 \in B$

it is a function



$y_1 = y_2$

it is a function



$y_1 \neq y_2$

it is not a function

Things required to establish a function

-A,B

-the rule whereby each member of A is associated with exactly one member of B.

Functions can be defined by

- -table,
- -graph,
- -by instructions. Most instructions are given in mathematical formulas.

Mathematical formulas can be

- -explicit (the dependent variable is expressed) or
- -implicit (the dependent variable is not expressed).

Domain of a function

$f: A \longrightarrow B$ ($\exists$ =exists, exist)

$$D_f = \{x \in A \mid \exists\, y \in B \text{ so that } f(x) = y\}$$

Range of a function

$f: A \longrightarrow B$

$$R_f = \{y \in B \mid \exists\, x \in D_f \text{ so that } f(x) = y\}$$

x - is the independent variable or argument of the function,

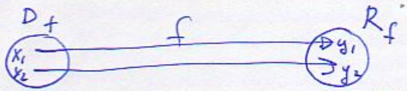
y - is the dependent variable or the function value.

So we can say by function $f\ A \rightarrow B$ each element of the domain (D_f) is mapped into a member of the range (R_f).

$$\underset{x \in D_f}{x} \xrightarrow[y = f(x)]{f} \underset{y \in R_f}{y}$$

D_f R_f

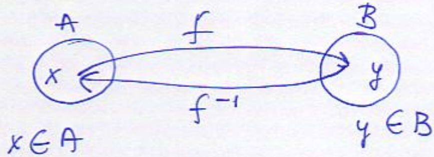
D A function is invertible if there is a one-to-one mapping.



invertible function

not invertible function

f: A $\longrightarrow$ B, f is an invertible function



f^{-1} is the inverse of the function f

F4

$$f: D_f \longrightarrow R_f$$

$$f^{-1}: R_f \longrightarrow D_f$$

$x \in D_f$

$$y = f(x)$$

$$f^{-1}(y) = f^{-1}(f(x)) = x$$

$y \in R_f$

$$x = f^{-1}(y)$$

$$f(x) = f(f^{-1}(y)) = y$$

<u>How to invert a function?</u> $y = f(x)$

-express x. This is the inverse, but x is the dependent variable and y is the independent one,

-interchange x with y.

$f : \mathbf{R} \rightarrow \mathbf{R}$ real-real function

function features

D $f : \mathbb{R} \rightarrow \mathbb{R}$

is monotone increasing when for $\forall\, x_1, x_2 \in D_f$

if $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2) \quad \nearrow$

is strictly monotone increasing

if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \quad \uparrow$

is monotone degreasing

if $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2) \quad \searrow$

is strictly monotone degreasing

if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \quad \downarrow$

T Strictly monotone functions are invertible ones.

D f : **R** $\longrightarrow$ **R** is an even function if for $\forall x \in D_f$

$$f(x) = f(-x)$$

is an **odd function** if for $\forall x \in D_f$

$$f(x) = -f(-x)$$

D f : **R** $\longrightarrow$ **R** is a periodic function with period $P \in \mathbb{R}$

if for $\forall x \in D_f$

$$f(x) = f(x+P)$$

D f: **R** $\longrightarrow$ **R** is a bounded function if there is a $K \in \mathbb{R}$

So that for $\forall x \in D_f$

$$|f(x)| \leq K$$

Operations with functions

1 $(f+g)(x) = f(x) + g(x)$

2 $(f-g)(x) = f(x) - g(x)$

3 $(f \cdot g)(x) = f(x) \cdot g(x)$

4 $\frac{f}{g}(x) = \frac{f(x)}{g(x)} \qquad g(x) \neq 0$

5 function inversion

6 compound function formation

D $g: A \longrightarrow B$
$f: C \longrightarrow D$ $\qquad C \supset B$

$$(f \circ g)(x) = f(g(x))$$

Example $g(x) = \sin x \,, \quad f(x) = \sqrt{1+x^2}$

$$f(g(x)) = \sqrt{1 + \sin^2 x}$$
$$g(f(x)) = \sin \sqrt{1+x^2}$$

D The basic functions are

$f_1 = c \qquad c \in \mathbf{R}$

$f_2 = x$

$f_3 = a^x \qquad a \in \mathbf{R}, \quad a > 1$

$f_4 = \sin x$

D The elementary functions are all functions constructed from the basic functions by the aid of the six function operations in a finite number of steps and can be expressed explicitly.

Elementary functions

1 Polynomials

$f: \mathbf{R} \longrightarrow \mathbf{R}$

$$P_n(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$$

$$a_1, a_2, \dots, a_{n-1}, a_n \in \mathbb{R}$$

$$n \in \mathbb{N}, \ n \neq 0$$

$$D_{P_n} = \mathbb{R} \qquad R_{P_n} = ?$$

T <u>The basic law of algebra</u>. P_n can be expressed as the product of linear terms.

$$P_n(x) = a_0(x - x_1)(x - x_2)\ldots\ldots(x - x_n)$$

$x_1, x_2, x_3, \ldots, x_n \in \mathbf{C}$ they are the complex roots of equation $P_n(x) = 0$

$P_n(y)$ can be expressed as the product of linear and quadratic terms.

2. Rational functions

$f: \mathbf{R} \longrightarrow \mathbf{R}$

$$R_{n,m}(x) = \frac{P_n(x)}{P_m(x)}$$

$$D_{R_{n,m}} = \mathbb{R} \setminus \{\text{the roots of } P_m(x)\}$$

$$R_{R_{n,m}} = ?$$

D $R_{n,m}$ is a proper rational function if $n < m$.

$$R_{n,m} = \frac{P_n}{P_m}$$

T Not proper rational functions can be changed to the sum of a proper rational function and a polynomial by division.

3. Irrational functions

Irrational functions are functions containing the inverse power functions, that is the independent variable x is raised to a not integer power.

Examples

$$f_1(x) = 5x^{1/2} - 3x^{1/3} = 5\sqrt{x} - 3\sqrt[3]{x}$$

$$f_2(x) = \frac{1}{\sqrt{x}+1}$$

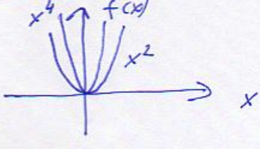
Power functions

$f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^n$

$n \in \mathbb{N},\ n \neq 0$

$$\left(f(x) = x^0 = 1 \right)$$

a. if n is an even integer

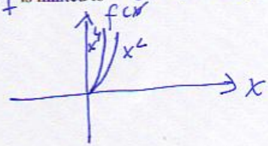


$D_f = \mathbb{R}$ $\quad$ $R_f = [0, +\infty)$

f(x) is not invertible

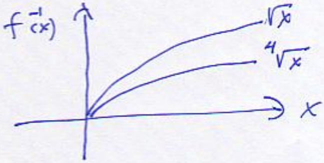
but f(x) : $[0, +\infty) \longrightarrow [0, +\infty)$

D_f is limited to $[0, +\infty)$

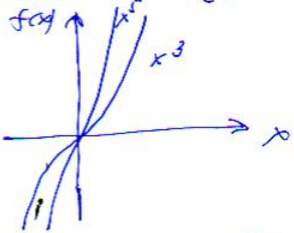


f is invertible.

$f^{-1}(x) = \sqrt[n]{x} = x^{\frac{1}{n}}$



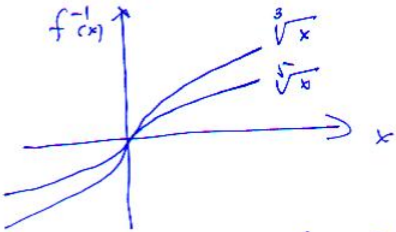
b, if n is an odd integer $(x = 1 \quad f(x) = x^1 = x)$



$D_f = R_f = \mathbb{R}$

The function is strictly monotone increasing $\Longrightarrow$ it is invertible.

$$f^{-1}(x) = \sqrt[n]{x} = x^{\frac{1}{n}}$$



$$D_{f^{-1}} = R_{f^{-1}} = \mathbb{R}$$

D_f of irrational functions is limited if the root exponent is even, the argument of the root must not be negative.

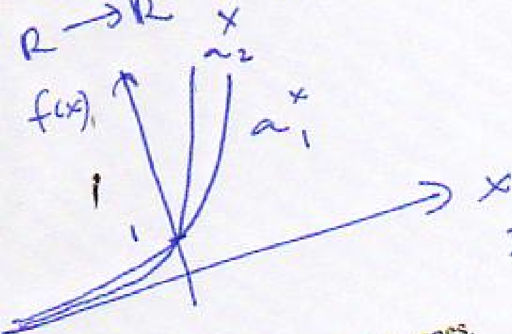
. Expo...

are functions containing exp...

Example $f(x) = a^x + \log_a x$ $\quad a \in R$

$f = a^x \quad a > 1$

$f: R \to R$



$a_2 > \ldots$

$D_f = R\ldots$

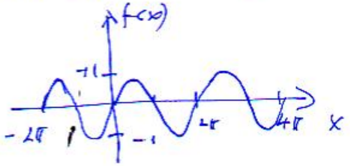
Exponential functions are invertible ones.

$f^{-1}(x) = \log_a x$

5. Trigonometric functions

are functions containing term originated of sin x.

$f : \mathbb{R} \to \mathbb{R}$ $\qquad f(x) = \sin x$



$D_f = \mathbb{R}$

$R_f = [-1, +1]$

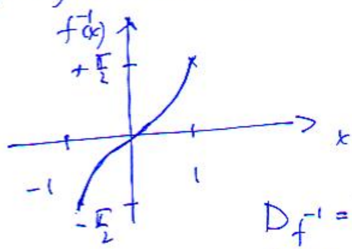
is bounded, odd, periodic and not invertible

But $f(x) = \sin x$ $\qquad D_f = \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$

$R_f = [-1, +1]$

is invertible.

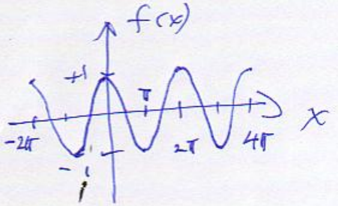
$f^{-1}(x) = \arcsin x$



$D_{f^{-1}} = [-1, +1]$

$R_{f^{-1}} = \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$

$f: \mathbb{R} \to \mathbb{R}$ $\qquad$ $f(x) = \cos x$



$D_f = \mathbb{R}$

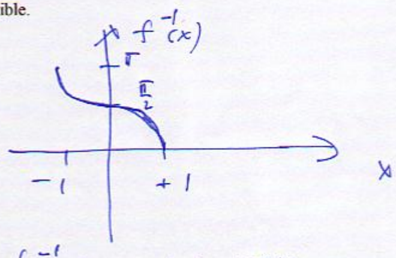
$R_f = [-1, +1]$

is bounded, even, periodic, not invertible.

But $\quad f(x) = \cos x \qquad D_f = [0, \pi]$

$R_f = [-1, +1]$

is invertible.

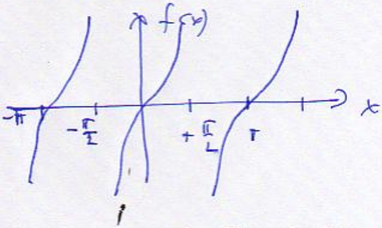


$f^{-1}(x) = \arccos x$

$D_{f^{-1}} = [-1, +1] \qquad R_{f^{-1}} = [0, \pi]$

F16

f : $\mathbb{R} \to \mathbb{R}$ $\qquad f(x) = \tan x$



is odd, periodic, not invertible, not bounded,.

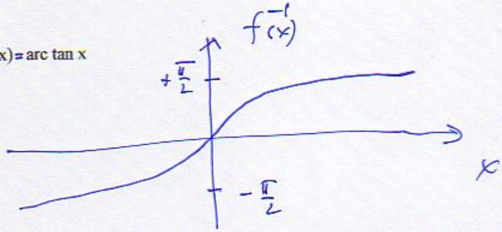
But $\qquad f(x) = \tan x$

$$D_f = \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right)$$

$$R_f = \mathbb{R}$$

is invertible.

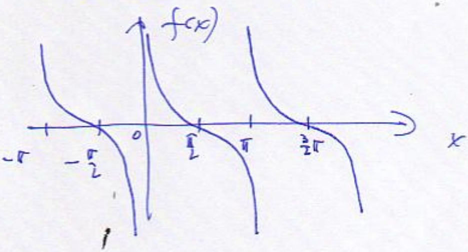
$f^{-1}(x) = \arctan x$



$D_{f^{-1}} = \mathbb{R}$, $\quad R_{f^{-1}} = \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right)$

F17

$f(x): R \to R$ $\quad f(x) = \text{ctan } x$

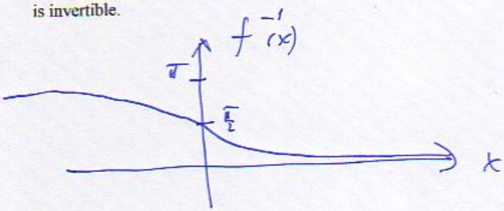


Is odd, periodic, not bounded, not invertible.

But $f(x) = \text{ctan } x$ $\quad D_f = (0, \pi)$

$R_f = \mathbb{R}$

is invertible.



$D_{f^{-1}} = R$

$R_{f^{-1}} = (0, \pi)$

Appendix

Special number sets, intervals

Closed interval

$$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

Open interval

$$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$$

$$(a,b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$

$$[a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$

$$(-\infty,b] = \{x \in \mathbb{R} \mid x \leq b\}$$

$$(-\infty,b) = \{x \in \mathbb{R} \mid x < b\}$$

$$[a,+\infty) = \{x \in \mathbb{R} \mid a \leq x\}$$

$$(a,+\infty) = \{x \in \mathbb{R} \mid a < x\}$$

$$(-\infty,+\infty) = \mathbb{R}$$